Identifying and Modeling Autoregressive Conditional Heteroskedastic Processes

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Worcester Polytechnic Institute

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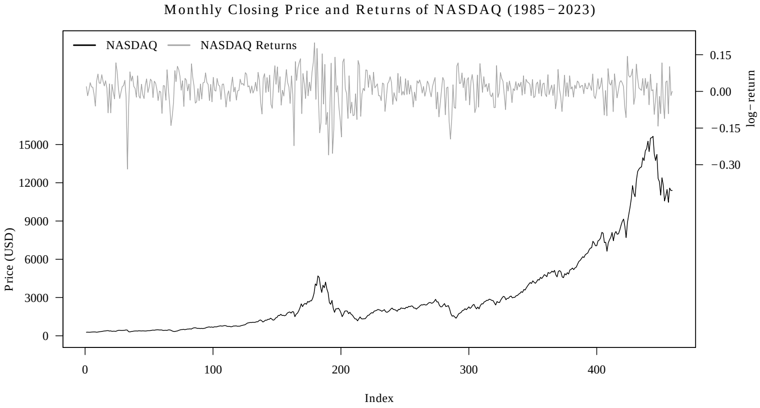
Introduction

Motivation



Asset behavior

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  - intuitively, periods of high fluctuation appear to be less predictable
  - volatile $\implies$ uncertain?
- **TLDR; it is important to model risk!**

How do we quantify risk?

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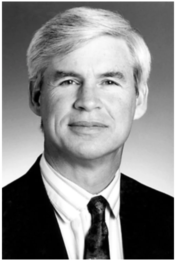
- Many models have been presented
- Variance found to be strong indicator
  - CAPM Model (Sharpe, 1964)
  - Black-Scholes Model (Black & Scholes, 1973)
- **volatility** of an asset commonly expressed through the std of the series

Can we use a time series to model volatility?



Robert Engle

- Recall, we model volatility through std/variance

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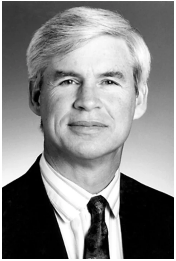
Can we use a time series to model volatility?

- Usually, we assume constant unconditional variance over a period
  - realized volatility
  - for stationarity



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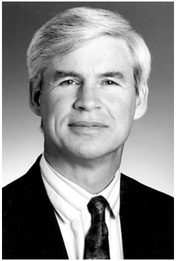
- Actual volatility is latent, not directly observable
  - **Main Observation:** volatility is constantly fluctuating



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Can we use a time series to model volatility?

- Usually, we assume constant unconditional variance over a period
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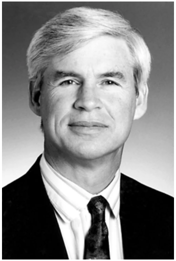
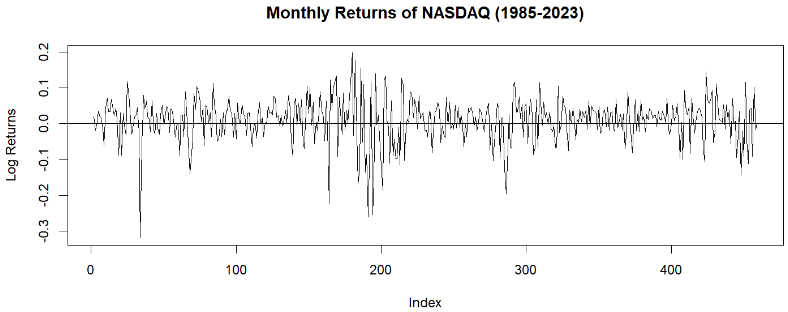
- In the seminal work of Engle (1982)
  - model volatility as a time-varying process
  - modeling of residual series
- Recall, we model volatility through std/variance



Robert Engle

Our Data



Monthly Returns of NASDAQ (1985-2023)
Log Returns
0.2
0.1
0.0
-0.1
-0.2
-0.3
0
100
200
300
400
Index

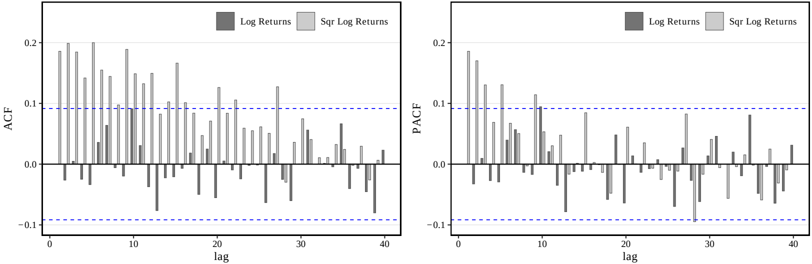
Building an ARCH-type model for stationary time series X_t:

1. Break into mean and innovation (residual) processes

$$X_t = \mu_t + \epsilon_t$$

2. Removal of mean process μ_t through ARMA process
3. Identify ARCH process in residual series
4. Selection of ARCH-model type
5. Model construction and joint parameter estimation

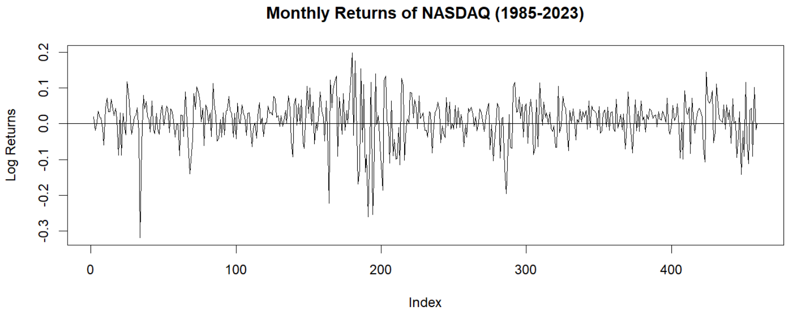
Identifying Mean Process



We opt not to break this process into mean/innovation.[1]

[1] We also notice from the ACF that we do not need to remove the mean process in this data set. Already no serial correlation and stationary (see report for further details).

Our Data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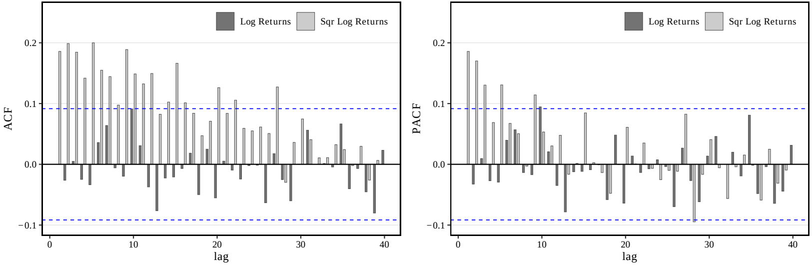
Properties we look for in an ARCH process:

1. Unpredictability/conditional heteroskedasticity
2. Volatility Clustering
3. Leptokurtic

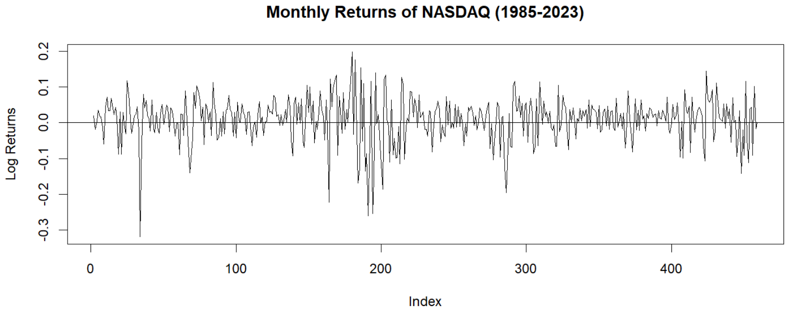
Classification of ARCH-type Models

Unpredictability



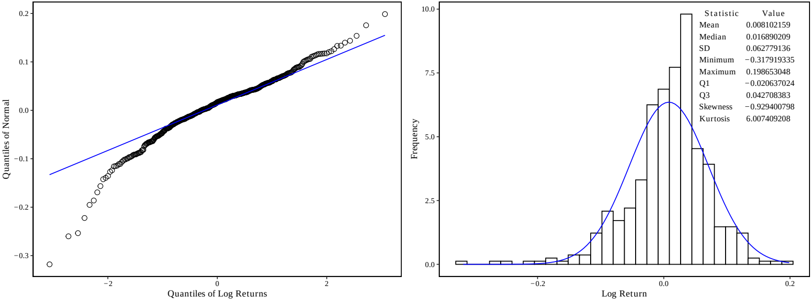
Classification of ARCH-type Models

Volatility Clustering



Classification of ARCH-type Models

Leptokurticity



ARCH Model

Consider (residual) series $\{\epsilon_t\}$ exhibiting ARCH-process behavior.

Definition

An autoregressive conditional heteroskedasticity (ARCH(m)) model expresses the variance σ^2 of ϵ_t as an AR process of ϵ_t^2:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \epsilon_{t-i}^2$$

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Definition

A Generalized-ARCH (GARCH(m, s)) model is one of the form expresses the variance σ^2 of ϵ_t as an ARMA process of ϵ_t^2 and σ_t^2:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{s} \beta_i \sigma_{t-i}^2$$

ARCH Models

strengths and weaknesses

Why or why not GARCH and ARCH models?

- Symmetric Volatility Shocks
- Conditional Heteroskedasticity
- Aforementioned patterns are modeled

ARCH Model

Parameter selection

- Several different methods to select order of (G)ARCH model
  - For big enough samples, PACF of the squared ARCH-series is enough to estimate ARCH order
  - GARCH(1,1) and other low-order models have been experimentally found to frequently outperform larger-order GARCH models (Jafari et al., 2007)
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- Parameter estimation and likelihood function depends on the assumed distribution of residuals of (G)ARCH model
  - Will not discuss in this talk

Model Diagnostics and Forecasting

We compare performance of ARCH(1), ARCH(2), ARCH(3), GARCH(1,1), GARCH(1,2), GARCH(2,1) models:[2]

	AIC	BIC
ARCH(1)	-2.795142	-2.768110
ARCH(2)	-2.826293	-2.790251
ARCH(3)	-2.850076	-2.805023
GARCH(1,1)	-2.864962	-2.828919
GARCH(1,2)*	-2.859456	-2.814403
GARCH(2,1)*	-2.859487	-2.814434

Table: Model comparison on NASDAQ historic monthly log returns. All models are fitted under normal standardized residuals.

*Some parameters are found to be insignificant. We leave them in the model for the moment.

[2] Note that for this sample data we do not have any autocorrelation and thus do not need the ARMA piece of the final joint parameter estimation. Thus all candidate models are ARMA(0,0)-(G)ARCH.

Fitted Model and Forecasting

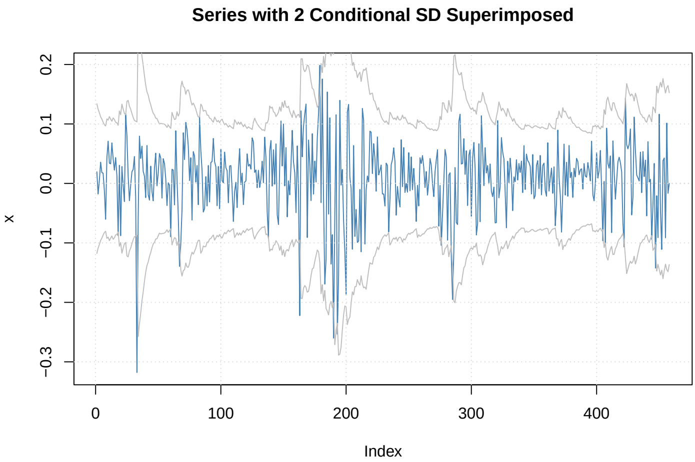

Figure: Two conditional SDs of fitted GARCH(1,1) model.

Fitted Model and Forecasting

Our model:

$$\mu = 0.00964380$$

and

$$\sigma_t^2 = 0.00025769 + 0.14500309\epsilon_{t-1}^2 + 0.79402682\sigma_{t-1}^2$$

Fitted Model and Forecasting

	mean	meanError	sd	lower	upper
1	0.0096	0.0665	0.0665	-0.1206	0.1399
2	0.0096	0.0664	0.0664	-0.1204	0.1397
3	0.0096	0.0663	0.0663	-0.1203	0.1396
4	0.0096	0.0662	0.0662	-0.1201	0.1394
5	0.0096	0.0661	0.0661	-0.1200	0.1393

Table: GARCH(1,1) forecast of monthly NASDAQ log-returns for next 5 time steps (summary of R output)

Any questions?

Bibliography

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*(3), 637–654. https://doi.org/10.1086/260062

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, *50*(4), 987. https://doi.org/10.2307/1912773

Jafari, G. R., BAHRAMINASAB, A., & NOROUZZADEH, P. (2007). WHY DOES THE STANDARD GARCH(1, 1) MODEL WORK WELL? *International Journal of Modern Physics C*, *18*(07), 1223–1230. https://doi.org/10.1142/s0129183107011261

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance*, *19*(3), 425. https://doi.org/10.2307/2977928